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On the Geometry of Semi-Slant Submanifolds in Kenmotsu Manifolds endowed with the Schouten-van Kampen Connection

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Abstract. The main objective of this paper is to investigate the geometry of semi-slant submanifolds in Kenmotsu manifolds defined with respect to the Schouten-van Kampen connection. Unlike the Levi-Civita connection, the Schouten-van Kampen connection offers a more rigid geometric framework in which the structure vector field ξ and the metric tensor remain parallel. In this study, we first establish the existence of semi-slant submanifolds in this setting and obtain necessary and sufficient conditions for their characterization. Furthermore, we investigate the integrability conditions of the invariant and slant distributions associated with the semi-slant structure, and establish new results on the total geodesicity of these submanifolds as well as on the geometric properties of the induced connection. Finally, we provide a non-trivial example of a 5-dimensional semi-slant submanifold in $\mathbb{R}^7$ to illustrate the theoretical results obtained.

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1. Introduction

The study of almost contact metric manifolds has long been a central topic of differential geometry, providing a rich framework for examining diverse geometric structures. Tanno's classification of connected almost contact metric manifolds, as described in [21], divides these manifolds into three distinct categories. One of these classes, characterized by the warped product space $\mathbb{R} \times_f \mathbb{C}^n$, was extensively studied by Kenmotsu [11], who established the fundamental tensor equations defining what are now called Kenmotsu manifolds. Unlike Sasakian manifolds, Kenmotsu manifolds are not K-contact, yet they possess a unique geometric richness that has attracted significant attention in the study of submanifold theory, especially regarding curvature properties and structural classifications [9, 12].

In the field of submanifold geometry, B.Y. Chen's introduction of slant immersions [4] marked a turning point, generalizing both holomorphic and totally real immersions in complex geometry. Chen and Tazawa [7] provided explicit examples of slant immersions in complex Euclidean spaces $\mathbb{C}^2$ and $\mathbb{C}^4$. Concurrently, slant immersions of Kaehler $\mathbb{C}$-spaces into complex projective spaces were analyzed by Maeda, Ohnita, and Udagawa [15]. Shifting the focus to almost contact metric geometry, A. Lotta [13] introduced the definition of slant immersions for Riemannian manifolds and proved several key properties associated with these immersions. Later, the study of slant submanifolds was specialized to Kenmotsu manifolds by R.S. Gupta et al. [9].

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Subsequently, N. Papaghiuc [17] defined the concept of semi-slant submanifolds, presenting them as a generalization of the established CR-submanifolds. The study of slant submanifold theory in Sasakian manifolds was carried out by J.L. Cabrerizo et al. [5]. Furthermore, V.A. Khan et al. [12] established fundamental geometric characterizations for slant and semi-slant submanifolds in Kenmotsu manifolds. Recently, the study of slant and semi-slant submanifolds has been extended to various geometric environments, such as Metallic Riemannian manifolds [10], Norden manifolds [1], and almost contact metric 3-structures [14].

While the majority of these studies utilize the Levi-Civita connection, the geometry of submanifolds can often be better understood through adapted connections that align with the manifold's distributions. The Schouten-van Kampen connection, originally introduced to study non-holonomic manifolds [18], provides such a framework. Unlike the Levi-Civita connection, the Schouten-van Kampen connection adapts to the pair of distributions on the manifold and preserves them under parallel transport. Solov'ev [19] and later Olszak [16] demonstrated the utility of this connection in characterizing hyperdistributions and almost contact metric structures. In the context of Kenmotsu manifolds, this connection offers a more rigid geometric constraint than the Levi-Civita connection, potentially revealing new integrability conditions and curvature properties that remain hidden under the standard metric connection [22, 23, 25].

Despite the extensive literature on slant and semi-slant submanifolds in Kenmotsu manifolds [9,24], the investigation of these structures with respect to the Schouten-van Kampen connection remains an underexplored area. Most existing results, such as those regarding the parallelism of the shape operator Q or the integrability of distributions, rely heavily on the symmetry of the Levi-Civita connection. Introducing the Schouten-van Kampen connection alters the Gauss and Weingarten formulas, specifically regarding the torsion tensor and the behavior of the second fundamental form, thereby necessitating a re-examination of the fundamental geometric characterizations.

Motivated by these developments, this study aims to bridge this gap by investigating semi-slant submanifolds in Kenmotsu manifolds endowed with the Schouten-van Kampen connection. We investigate the geometric properties of such manifolds, derive new integrability conditions for the invariant and slant distributions and establish necessary and sufficient conditions for these submanifolds to be totally geodesic. Furthermore, we obtain characterization theorems involving the shape operator and the slant angle, contrasting our findings with those obtained under the Levi-Civita connection. Finally, to demonstrate the existence and non-triviality of such structures, we construct a concrete example of a 5-dimensional semi-slant submanifold in $\mathbb{R}^7$.

2. Preliminaries

This section provides a brief overview of the fundamental definitions and formulas related to Kenmotsu manifolds, the Schouten-van Kampen connection, and semi-slant submanifolds which will be used throughout the paper.

Consider an odd-dimensional differentiable manifold $\bar{\mathcal{M}}$ of dimension $(2n+1)$ equipped with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$-type tensor field, ξ is the structure vector field, η is a 1-form and g is a Riemannian metric such that

$$\phi^2(\mathcal{U}) = -\mathcal{U} + \eta(\mathcal{U})\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0,$$
$$g(\phi\mathcal{U}, \phi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) - \eta(\mathcal{U})\eta(\mathcal{V}),$$
$$g(\mathcal{U}, \phi\mathcal{V}) = -g(\phi\mathcal{U}, \mathcal{V}), \quad g(\mathcal{U}, \xi) = \eta(\mathcal{U}),$$

for any vector fields $\mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{T}\bar{\mathcal{M}})$ [2]. If this structure additionally satisfies the condition,

$$(\bar{\nabla}_{\mathcal{U}}\phi)\mathcal{V} = g(\phi\mathcal{U}, \mathcal{V})\xi - \eta(\mathcal{V})\phi\mathcal{U},$$

then $\bar{\mathcal{M}}$ is called a Kenmotsu manifold, where $\bar{\nabla}$ denotes the Levi-Civita connection of g. Furthermore, on a Kenmotsu manifold, the following relations hold [11]

$$\bar{\nabla}_{\mathcal{U}}\xi = \mathcal{U} - \eta(\mathcal{U})\xi, \tag{2.1}$$
$$\left(\bar{\nabla}_{\mathcal{U}}\eta\right)\mathcal{V} = g(\mathcal{U}, \mathcal{V}) - \eta(\mathcal{U})\eta(\mathcal{V}). \tag{2.2}$$

Now, let $\mathcal{M}$ be a submanifold immersed in $\bar{\mathcal{M}}$. We denote the Lie algebra of vector fields on $\mathcal{M}$ by $\mathcal{T}\mathcal{M}$ and the set of all vector fields normal to $\mathcal{M}$ by $\mathcal{T}^{\perp}\mathcal{M}$. Let ∇ be the induced Riemannian connection on $\mathcal{M}$. Then, structural equations of Gauss and Weingarten are expressed as follows:

$$\bar{\nabla}_{\mathcal{U}}\mathcal{V} = \nabla_{\mathcal{U}}\mathcal{V} + \sigma(\mathcal{U}, \mathcal{V}), \quad \bar{\nabla}_{\mathcal{U}}\mathcal{W} = -A_{\mathcal{W}}\mathcal{U} + \nabla^{\perp}_{\mathcal{U}}\mathcal{W}, \tag{2.3}$$

for all vector fields $\mathcal{U}, \mathcal{V} \in \mathcal{TM}$ and any $\mathcal{W} \in \mathcal{T}^{\perp}\mathcal{M}$. In these equations, $\nabla^{\perp}$ denotes the connection in the normal bundle, σ represents the second fundamental form, and $A_{\mathcal{W}}$ is the shape operator (Weingarten endomorphism) corresponding to the normal vector $\mathcal{W}$. The relationship between σ and $A_{\mathcal{W}}$ is given by

$$g(A_{\mathcal{W}}\mathcal{U}, \mathcal{V}) = g\left(\sigma\left(\mathcal{U}, \mathcal{V}\right), \mathcal{W}\right).$$

We note that the induced Riemannian metric on $\mathcal{M}$ is also denoted by g.

The mean curvature vector $\mathcal{H}$ is defined as $\mathcal{H} = (1/m)trace(\sigma)$, where m is the dimension of $\mathcal{M}$. The submanifold $\mathcal{M}$ is termed minimal if $\mathcal{H}$ vanishes identically.

For a linear connection ∇ on a manifold $\mathcal{M}$, the torsion tensor $\mathcal{T}$ is defined by

$$\mathcal{T}\left(\mathcal{U}, \mathcal{V}\right) = \nabla_{\mathcal{U}}\mathcal{V} - \nabla_{\mathcal{V}}\mathcal{U} - \left[\mathcal{U}, \mathcal{V}\right].$$

The connection is symmetric if $\mathcal{T}$ vanishes, otherwise it is non-symmetric. Specifically, ∇ is called a semi-symmetric connection if the torsion tensor satisfies

$$\mathcal{T}\left(\mathcal{U}, \mathcal{V}\right) = \eta\left(\mathcal{V}\right)\mathcal{U} - \eta\left(\mathcal{U}\right)\mathcal{V}.$$

For any $\mathcal{U} \in \mathcal{TM}$, we write

$$\phi\mathcal{U} = \mathcal{TU} + \mathcal{FU}, \tag{2.4}$$

where $\mathcal{TU}$ and $\mathcal{FU}$ denote the tangential and normal components of $\phi\mathcal{U}$, respectively. Similarly, for any normal field $\mathcal{W} \in \mathcal{T}^{\perp}\mathcal{M}$, the decomposition is given by

$$\phi\mathcal{W} = \mathcal{BW} + \mathcal{CW}, \tag{2.5}$$

with $\mathcal{BW}$ and $\mathcal{CW}$ representing the tangential and normal components of $\phi\mathcal{W}$, respectively.

The submanifold $\mathcal{M}$ is called invariant if $\mathcal{F}$ is identically zero, and anti-invariant if $\mathcal{T}$ is identically zero. If $\mathcal{T}$ is the endomorphism defined by (2.4), then

$$g(\mathcal{TU}, \mathcal{V}) + g(\mathcal{U}, \mathcal{TV}) = 0.$$

Utilizing the operator $\mathcal{Q} = \mathcal{T}^2$, the covariant derivatives of the tensor fields $\mathcal{Q}$, $\mathcal{T}$ and $\mathcal{F}$ are defined as

$$\begin{aligned}(\nabla_{\mathcal{U}}\mathcal{Q})\mathcal{V} &= \nabla_{\mathcal{U}}(\mathcal{QV}) - \mathcal{Q}(\nabla_{\mathcal{U}}\mathcal{V}),\\ (\nabla_{\mathcal{U}}\,\mathcal{T})\mathcal{V} &= \nabla_{\mathcal{U}}(\mathcal{TV}) - \mathcal{T}(\nabla_{\mathcal{U}}\mathcal{V}),\end{aligned} \tag{2.6}$$

and

$$(\nabla_{\mathcal{U}}\mathcal{F})\mathcal{V} = \nabla^{\perp}_{\mathcal{U}}(\mathcal{FV}) - \mathcal{F}(\nabla_{\mathcal{U}}\mathcal{V}), \tag{2.7}$$

for $\mathcal{U}, \mathcal{V} \in \mathcal{TM}$. Considering the properties of a Kenmotsu manifold, applying equations (2.1) and (2.3) yields

$$\nabla_{\mathcal{U}}\xi = \mathcal{U} - \eta(\mathcal{U})\xi, \tag{2.8}$$

and

$$\sigma(\mathcal{U}, \xi) = 0, \tag{2.9}$$

for each $\mathcal{U} \in \mathcal{TM}$. Moreover, from (2.9)

$$A_{\mathcal{W}}\xi = 0, \qquad \eta\left(A_{\mathcal{W}}\mathcal{U}\right) = 0,$$

for each $\mathcal{W} \in \mathcal{T}^{\perp}\mathcal{M}$.

By combining the Gauss and Weingarten formulas with (2.4), (2.5), (2.6) and (2.7) imply

$$\begin{aligned}(\nabla_{\mathcal{U}}\,\mathcal{T})\mathcal{V} &= A_{\mathcal{FV}}\mathcal{U} + \mathcal{B}\sigma(\mathcal{U}, \mathcal{V}) - g(\mathcal{U}, \mathcal{TV})\xi - \eta(\mathcal{V})\mathcal{TU},\\ (\nabla_{\mathcal{U}}\mathcal{F})\mathcal{V} &= \mathcal{C}\sigma(\mathcal{U}, \mathcal{V}) - \sigma(\mathcal{U}, \mathcal{TV}) - \eta(\mathcal{V})\mathcal{FU},\end{aligned} \tag{2.10}$$

for any $\mathcal{U}, \mathcal{V} \in \mathcal{TM}$.

By using relations (2.12) and (2.10), we can establish necessary and sufficient conditions for the commutativity of the shape operators.

Specifically, the validity of equation (2.12) is equivalent to the condition

$$A_{\mathcal{FV}}\mathcal{U} = A_{\mathcal{FU}}\mathcal{V} \tag{2.11}$$

if and only if

$$(\nabla_{\mathcal{U}}\,\mathcal{T})\mathcal{V} = g(\mathcal{TU}, \mathcal{V})\xi - \eta(\mathcal{V})\mathcal{TU}. \tag{2.12}$$

Using (2.10), a similar calculation shows that

$$(\nabla_{\mathcal{U}}\mathcal{F})\mathcal{V} = -\eta(\mathcal{V})\mathcal{F}\mathcal{U},$$

if and only if

$$A_{\mathcal{F}}\mathcal{T}\mathcal{V} = -A_{C\mathcal{F}}\mathcal{V},$$

for any $\mathcal{U}, \mathcal{V} \in \mathcal{T}\mathcal{M}$ and $\mathcal{F} \in \mathcal{T}^{\perp}\mathcal{M}$.

3. Slant Immersions

In this section, we review the notion of slant submanifolds and provide a characterization theorem which will be fundamental for the subsequent analysis with respect to the Schouten-van Kampen connection.

Let us consider $\mathcal{M}$ as a Riemannian submanifold isometrically immersed in an almost contact metric manifold denoted by $(\bar{\mathcal{M}}, \phi, \xi, \eta, g)$. Throughout this study, we assume that the structure vector field ξ remains tangent to the submanifold $\mathcal{M}$. Under this assumption, the tangent bundle $\mathcal{T}\mathcal{M}$ admits an orthogonal decomposition:

$$\mathcal{T}\mathcal{M} = \mathcal{D} \oplus \langle \xi \rangle ,$$

where $\mathcal{D}$ denotes the distribution orthogonal to ξ (i.e., $\mathcal{D}$= ker(η)).

For any non-zero vector $\mathcal{U} \in \Gamma(\mathcal{T}\mathcal{M})$ at a point $u \in \mathcal{M}$, which is linearly independent of ξ_u, the Wirtinger angle $\theta(\mathcal{U}) \in [0, \pi/2]$ is defined as the angle between the vector $\phi\mathcal{U}$ and the tangent space $\mathcal{T}_u\mathcal{M}$. The submanifold $\mathcal{M}$ is classified as a slant submanifold if this angle $\theta(\mathcal{U})$ is constant, meaning it is independent of both the choice of point $u \in \mathcal{M}$ and the vector $\mathcal{U} \in \mathcal{T}_u\mathcal{M} - < \xi_u >$ .This constant θ is referred to as the slant angle.

Special cases arise depending on the value of the slant angle θ. Invariant submanifolds correspond to $\theta = 0$, while anti-invariant submanifolds correspond to $\theta = \pi/2$. If the slant angle satisfies $0 < \theta < \pi/2$, then the submanifold is called a proper slant submanifold [13].

Theorem 3.1 ([5]). *Consider a submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ such that ξ is tangent to $\mathcal{M}$. $\mathcal{M}$ is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\mathcal{T}^2 = -\lambda I + \lambda\eta \otimes \xi.$$

Furthermore, in this case, the slant angle θ is related to λ by $\lambda = \cos^2\theta$.

Corollary 3.2. *Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $\bar{\mathcal{M}}$ with slant angle θ. Then the operator $\mathcal{Q}$, restricted to the distribution $\mathcal{D}$, has a unique eigenvalue $\lambda_1 = -\cos^2\theta$ at every point.*

Theorem 3.3. *Let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ where $\xi \in \mathcal{T}\mathcal{M}$. Then, the condition $\nabla\mathcal{Q} = 0$ is satisfied if and only if $\mathcal{M}$ is an anti-invariant submanifold.*

To investigate 3-dimensional slant submanifolds in Kenmotsu manifolds, we use the following local frame characterization:

Lemma 3.4 ([9]). *Let $\mathcal{M}$ be a 3-dimensional submanifold of an almost contact metric manifold $\bar{\mathcal{M}}$ with $\xi \in \mathcal{T}\mathcal{M}$. In the neighborhood of any point, there exist vector fields e_1, e_2 such that $\{e_1, e_2, \xi\}$ forms a local orthonormal frame satisfying:*

$$\mathcal{T}_{e_1} = \lambda_1 e_2, \quad \mathcal{T}_{e_2} = -\lambda_1 e_1,$$

where λ_1 is a locally defined function. If $\mathcal{M}$ is a slant submanifold with slant angle θ, then $\lambda_1 = \cos\theta$.

Theorem 3.5 ([9]). *A 3-dimensional submanifold $\mathcal{M}$ of a Kenmotsu manifold $\bar{\mathcal{M}}$ with $\xi \in \mathcal{T}\mathcal{M}$ is a slant submanifold if and only if the tensor $\mathcal{T}$ satisfies*

$$(\nabla_{\mathcal{U}}\,\mathcal{T})\mathcal{V} = -\eta(\mathcal{V})\mathcal{T}\mathcal{U} + g(\mathcal{V}, \mathcal{T}\mathcal{U})\xi,$$

for all $\mathcal{U}, \mathcal{V} \in \mathcal{T}\mathcal{M}$.

Combining these results with the equivalence of (2.11) and (2.12), we obtain the following characterization based on the Weingarten map.

Corollary 3.6 ([9]). *A 3-dimensional submanifold $\mathcal{M}$ of a Kenmotsu manifold is slant if and only if*

$$A_{\mathcal{F}\mathcal{V}}\mathcal{U} = A_{\mathcal{F}\mathcal{U}}\mathcal{V},$$

holds for all tangent vector fields $\mathcal{U}, \mathcal{V}$.

The concept of semi-slant submanifolds was introduced by N. Papaghiuc [17] as a generalization of CR-submanifolds and slant submanifolds in almost Hermitian geometry. This notion was later extended to almost contact metric manifolds and Sasakian manifolds by Cabrerizo et al. [6], and to Kenmotsu manifolds by V.A. Khan et al. [12].

A submanifold $\mathcal{M}$ is defined as semi-slant if there exist two orthogonal differentiable distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

(1) $\mathcal{T}\mathcal{M} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \langle \xi \rangle$,
(2) $\mathcal{D}_1$ is invariant under ϕ (*i.e.*, $\phi(\mathcal{D}_1) = \mathcal{D}_1$),
(3) $\mathcal{D}_2$ is a slant distribution with a slant angle $\theta \neq 0$, that is, ϕS makes a constant angle with $\mathcal{T}\mathcal{M}$ for all nonzero $S \in \mathcal{D}_2$.

In the normal bundle $\mathcal{T}^{\perp}\mathcal{M}$, the orthogonal complement of $\mathcal{F}\mathcal{D}_2$ is an invariant subbundle denoted by μ. Thus, the decomposition of the normal bundle is given by [12]

$$\mathcal{T}^{\perp}\mathcal{M} = \mathcal{F}\,\mathcal{D}_2 \oplus \mu.$$

4. The Schouten-van Kampen connection

In this section, we define the Schouten-van Kampen connection on a differentiable manifold equipped with two complementary orthogonal distributions. We first present the general definition and then adapt it to the specific geometry of Kenmotsu manifolds.

Let $\mathcal{M}$ be a connected differentiable manifold of dimension $(2n+1)$ endowed with a Riemannian metric g, and let ∇ denote the Levi-Civita connection on $\mathcal{M}$. We assume that the tangent bundle $\mathcal{T}\mathcal{M}$ decomposes into two complementary orthogonal distributions $\mathcal{H}$ and $\mathcal{V}$, such that $\mathcal{T}\mathcal{M} = \mathcal{H} \oplus \mathcal{V}$.

Following the framework established in [3], we assume $\mathcal{V}$ is a one-dimensional distribution spanned by a unit vector field ξ, and $\mathcal{H}$ is a distribution of codimension one defined by a 1-form η such that $\eta(\xi) = 1$. Thus, we have

$$\mathcal{H} = \ker \eta \text{ and } \mathcal{V} = \langle \xi \rangle.$$

For any vector field $\mathcal{U} \in \mathcal{T}\mathcal{M}$, we denote by $\mathcal{U}^h$ and $\mathcal{U}^v$ the projections of $\mathcal{U}$ onto the distributions $\mathcal{H}$ and $\mathcal{V}$, respectively. These projections are given by

$$\mathcal{U}^h = \mathcal{U} - \eta(\mathcal{U})\xi, \quad \mathcal{U}^v = \eta(\mathcal{U})\xi. \tag{4.1}$$

The Schouten-van Kampen connection, denoted by $\tilde{\nabla}$, is defined as the connection adapted to the pair of distributions $(\mathcal{H}, V)$ such that it preserves the splitting. It is constructed from the Levi-Civita connection ∇ by the following formula

$$\tilde{\nabla}_{\mathcal{U}}\mathcal{V} = (\nabla_{\mathcal{U}}\mathcal{V}^h)^h + (\nabla_{\mathcal{U}}\mathcal{V}^v)^v, \tag{4.2}$$

for all vector fields $\mathcal{U}, \mathcal{V}$ on $\mathcal{M}$. Note that the condition (4.2) ensures the parallelism of the distributions $\mathcal{H}$ and $\mathcal{V}$ with respect to the Schouten-van Kampen connection $\tilde{\nabla}$.

Using the projection relations in (4.1), the terms can be expanded as [19]

$$(\nabla_{\mathcal{U}}\mathcal{V}^h)^h = \nabla_{\mathcal{U}}\mathcal{V} - \eta(\nabla_{\mathcal{U}}\mathcal{V})\xi - \eta(\mathcal{V})\nabla_{\mathcal{U}}\xi,$$
$$(\nabla_{\mathcal{U}}\mathcal{V}^v)^v = (\nabla_{\mathcal{U}}\eta)(\mathcal{V})\xi + \eta(\nabla_{\mathcal{U}}\mathcal{V})\xi.$$

Substituting these into the definition allows us to express the Schouten-van Kampen connection in terms of the Levi-Civita connection as follows

$$\tilde{\nabla}_{\mathcal{U}}\mathcal{V} = \nabla_{\mathcal{U}}\mathcal{V} - \eta(\mathcal{V})\nabla_{\mathcal{U}}\xi + (\nabla_{\mathcal{U}}\eta)(\mathcal{V})\xi. \tag{4.3}$$

Thus, the second fundamental form B and the torsion $\mathcal{T}$ of $\tilde{\nabla}$ are [19, 20]

$$B(\mathcal{U}, \mathcal{V}) = \eta(\mathcal{V})\nabla_{\mathcal{U}}\xi - (\nabla_{\mathcal{U}}\eta)(\mathcal{V})\xi,$$

and

$$\mathcal{T}(\mathcal{U}, \mathcal{V}) = \eta(\mathcal{U})\nabla_{\mathcal{V}}\xi - \eta(\mathcal{V})\nabla_{\mathcal{U}}\xi + 2d\eta(\mathcal{U}, \mathcal{V})\xi,$$

where $B = \nabla - \tilde{\nabla}$.

The Schouten-van Kampen connection (4.3) serves as a powerful tool for characterizing geometric objects associated with the distributions $\mathcal{H}$ and $\mathcal{V}$. Perhaps the most significant feature of this connection, as noted by Olszak [16], is that the metric g, the structure vector ξ and the 1-form η are all parallel with respect to $\bar{\nabla}$;

$$\bar{\nabla}\xi = 0, \bar{\nabla}g = 0, \bar{\nabla}\eta = 0.$$

5. Basic results

In this section, we investigate the geometry of submanifolds immersed in a Kenmotsu manifold endowed with the Schouten-van Kampen connection. We establish the relation between the induced connections and determine the properties of the induced structure tensors.

Let $(\bar{\mathcal{M}}, \phi, \xi, \eta, g)$ be a Kenmotsu manifold with respect to the Schouten-van Kampen connection. Then using (2.1) and (2.2) in (4.3), we get

$$\widetilde{\bar{\nabla}}_{\mathcal{U}}\mathcal{V} = \bar{\nabla}_{\mathcal{U}}\mathcal{V} + g(\mathcal{U},\mathcal{V})\xi - \eta(\mathcal{V})\mathcal{U}. \tag{5.1}$$

Let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ and $\widetilde{\bar{\nabla}}$ be the Schouten-van Kampen connection on $\bar{\mathcal{M}}$. Then the Gauss formula is given by

$$\widetilde{\bar{\nabla}}_{\mathcal{U}}\mathcal{V} = \overset{*}{\nabla}_{\mathcal{U}}\mathcal{V} + m(\mathcal{U},\mathcal{V}),$$

where $\overset{*}{\nabla}$ is an induced connection on $\mathcal{M}$, and m is a $(0,2)$-type tensor field. From equations (2.3) and (5.1), we obtain

$$\overset{*}{\nabla}_{\mathcal{U}}\mathcal{V} + m(\mathcal{U},\mathcal{V}) = \nabla_{\mathcal{U}}\mathcal{V} + \sigma(\mathcal{U},\mathcal{V}) + g(\mathcal{U},\mathcal{V})\xi - \eta(\mathcal{V})\mathcal{U},$$

and

$$\begin{aligned} \overset{*}{\nabla}_{\mathcal{U}}\mathcal{V} &= \nabla_{\mathcal{U}}\mathcal{V} + g(\mathcal{U},\mathcal{V})\xi - \eta(\mathcal{V})\mathcal{U}, \\ m(\mathcal{U},\mathcal{V}) &= \sigma(\mathcal{U},\mathcal{V}). \end{aligned} \tag{5.2}$$

Moreover from equations (2.3) and (5.2), we have

$$\begin{aligned} \overset{*}{\nabla}_{\mathcal{U}}\mathcal{W} &= \nabla_{\mathcal{U}}\mathcal{W} + g(\mathcal{U},\mathcal{W})\xi - \eta(\mathcal{W})\mathcal{U}, \\ &= -A_{w}\mathcal{U} - \eta(\mathcal{W})\mathcal{U}. \end{aligned}$$

Now, Gauss and Weingarten formulas for semi-slant submanifolds of a Kenmotsu manifold with the Schouten-van Kampen connection are given by

$$\widetilde{\bar{\nabla}}_{\mathcal{U}}\mathcal{V} = \overset{*}{\nabla}_{\mathcal{U}}\mathcal{V} + \sigma(\mathcal{U},\mathcal{V}), \tag{5.3}$$

$$\widetilde{\bar{\nabla}}_{\mathcal{U}}\mathcal{W} = -A_{\mathcal{W}}\mathcal{U} - \eta(\mathcal{W})\mathcal{U} + \overset{*}{\nabla}^{\perp}_{\mathcal{U}}\mathcal{W}, \tag{5.4}$$

where σ is the second fundamental form of $\mathcal{M}$ and $A_{\mathcal{W}}$ is the Weingarten endomorphism associated with $\mathcal{W}$. The second fundamental form σ and the shape operator A are related by

$$g(\sigma(\mathcal{U},\mathcal{V}),\mathcal{W}) = g(-A_{\mathcal{W}}\mathcal{U} - \eta(\mathcal{W})\mathcal{U},\mathcal{V}).$$

Let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ with Schouten-van Kampen connection, we have

i: $\sigma(\mathcal{U},\phi\mathcal{V}) = \sigma(\phi\mathcal{U},\mathcal{V}) = \phi\,\sigma(\mathcal{U},\mathcal{V})$,

ii: $\phi A_{\mathcal{W}}\mathcal{U} = -A_{\mathcal{W}}\phi\,\mathcal{U} = A_{n\mathcal{W}}\mathcal{U}$,

for all $\mathcal{U},\mathcal{V} \in \chi(\mathcal{M})$ and $\mathcal{W} \in T^{\perp}(\mathcal{M})$.

Corollary 5.1. *The induced connection on semi-slant submanifolds of a Kenmotsu manifold with Schouten-van Kampen connection is a semi-symmetric metric connection.*

Proposition 5.2. *Let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ with Schouten-van Kampen connection. Then, for all $\mathcal{U},\mathcal{V} \in \mathcal{TM}$*

$$(\overset{*}{\nabla}_{\mathcal{U}}\mathcal{B})\mathcal{V} = A_{\mathcal{F}\mathcal{V}}\mathcal{U} - \mathcal{B}\sigma(\mathcal{U},\mathcal{V}),$$

$$(\overset{*}{\nabla}_{\mathcal{U}}\mathcal{F})\mathcal{V} = \mathcal{F}\sigma(\mathcal{U},\mathcal{V}) - \sigma(\mathcal{U},\mathcal{B}\mathcal{V}).$$

Proof. By using (2.4), (2.5), (5.3) and (5.4), we can show easily. □

Corollary 5.3. *Let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ with Schouten-van Kampen connection. Then, for all $\mathcal{U} \in \mathcal{T}\mathcal{M}$ and $V \in \mathcal{T}^{\perp}\mathcal{M}$, we get*

$$(\overset{*}{\nabla}_{\mathcal{U}}\mathcal{B})V = g(\mathcal{F}\mathcal{U}, V)\xi + g(\mathcal{U},\ \mathcal{B}V)\xi + A_{\mathcal{F}V}\mathcal{U} -\ \mathcal{B}A_V\mathcal{U},$$
$$(\overset{*}{\nabla}_{\mathcal{U}}\mathcal{F})V = -\sigma(\mathcal{U},\ \mathcal{B}V) - \mathcal{F}A_V\mathcal{U}.$$

Proof. By using (2.5), (5.1), (5.3) and (5.4) we can show easily. □

6. Integrability of Distributions on a Semi-slant Submanifold of a Kenmotsu Manifold with the Schouten-van Kampen Connection

Theorem 6.1. *Let $\mathcal{M}$ be a semi-slant submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ endowed with the Schouten-van Kampen connection. Then, for $\mathcal{U} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Z} \in \Gamma(\mathcal{D}_2)$, we have*

$$[\mathcal{U}, \xi] \in \Gamma(\mathcal{D}_1 \oplus \langle\xi\rangle) \text{ and } [\mathcal{Z}, \xi] \in \Gamma(\mathcal{D}_2 \oplus \langle\xi\rangle).$$

Proof. We have for all $\mathcal{U} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Z} \in \Gamma(\mathcal{D}_2)$, we get

$$g([\mathcal{U}, \xi], \mathcal{V}) = g(\overline{\nabla}_{\mathcal{U}}\xi, \mathcal{V}) - g(\overline{\nabla}_{\xi}\mathcal{U}, \mathcal{V}).$$

Using equations (5.2), (5.3), (5.4), we have

$$g([\mathcal{U}, \xi], \mathcal{V}) = -g(\nabla_{\xi}\mathcal{U}, \mathcal{V}).$$

We know that $g([\mathcal{U}, \xi], \mathcal{V}) = g(\nabla_{\xi}\mathcal{V}, \mathcal{U})$. Since $\mathcal{U} \in \Gamma(\mathcal{D}_1)$, we can write $\mathcal{U} = \phi\mathcal{Z}$ for any $\mathcal{Z} \in \Gamma(\mathcal{D}_1)$. Thus,

$$(g([\mathcal{U}, \xi], \mathcal{V}) = g(\mathcal{U}, \nabla_{\xi}\mathcal{V}) = g(\phi\mathcal{Z}, \nabla_{\xi}\mathcal{V}) = g(\sigma(\xi, \mathcal{Z}), \phi\mathcal{V}) = 0,$$

where $g(\sigma(\mathcal{U}, \mathcal{V}), \phi\mathcal{Z}) = g(\nabla_{\mathcal{U}}\mathcal{Z}, \phi\mathcal{V})$. Because of $\mathcal{V} \in \Gamma(\mathcal{D}_2)$ we have $[\mathcal{U}, \xi] \in \Gamma(\mathcal{D}_1 \oplus \langle\xi\rangle)$.

Also for $\mathcal{Z} \in \Gamma(\mathcal{D}_2)$ and $\mathcal{V} \in \Gamma(\mathcal{D}_1)$,

$$g([\mathcal{Z}, \xi], \mathcal{V}) = g(\overline{\nabla}_{\mathcal{Z}}\xi, \mathcal{V}) - g(\overline{\nabla}_{\xi}\mathcal{Z}, \mathcal{V}).$$

Using equations (5.2), (5.3), (5.4), we have

$$g([\mathcal{Z}, \xi], \mathcal{V}) = -g(\nabla_{\xi}\mathcal{Z}, \mathcal{V}).$$

Since $\mathcal{V} \in \Gamma(\mathcal{D}_1)$, then we can write $\mathcal{V} = \phi\mathcal{U}$ for any $\mathcal{U} \in \Gamma(\mathcal{D}_1)$. Thus, we have $(g([\mathcal{Z}, \xi], \mathcal{V}) = g(\mathcal{Z}, \nabla_{\xi}\mathcal{V}) = g(\mathcal{Z}, \phi\nabla_{\xi}\mathcal{V}) = 0$. Hence, if $\mathcal{V} \in \Gamma(\mathcal{D}_1)$ and $g([\mathcal{Z}, \xi], \mathcal{V}) = 0$ then, we get $[\mathcal{Z}, \xi] \in \Gamma(\mathcal{D}_2 \oplus \langle\xi\rangle)$. □

Theorem 6.2. *Let $\mathcal{M}$ be a semi-slant submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ endowed with Schouten-van Kampen connection. Then, the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2$ is integrable.*

Proof. For all $\mathcal{U}, \mathcal{V} \in \mathcal{D}_1 \oplus \mathcal{D}_2$, we have

$$g([\mathcal{U}, \mathcal{V}], \xi) = g(\overline{\nabla}_{\mathcal{U}}\mathcal{V}, \xi) - g(\overline{\nabla}_{\mathcal{V}}\mathcal{U}, \xi).$$

Using (5.1)

$$g([\mathcal{U}, \mathcal{V}], \xi) = g(\overset{\approx}{\overline{\nabla}}_{\mathcal{U}}\mathcal{V} - g(\mathcal{U}, \mathcal{V})\xi + \eta(\mathcal{U})\mathcal{V}, \xi) - g(\overset{\approx}{\overline{\nabla}}_{\mathcal{V}}\mathcal{U} - g(\mathcal{V}, \mathcal{U})\xi + \eta(\mathcal{V})\mathcal{U}, \xi),$$
$$= -g(\mathcal{V}, \overset{\approx}{\overline{\nabla}}_{\mathcal{U}}\xi) + g(\mathcal{U}, \overset{\approx}{\overline{\nabla}}_{\mathcal{V}}\xi) = 0.$$

Thus, we get $\eta([\mathcal{U}, \mathcal{V}]) = 0$. Then, $[\mathcal{U}, \mathcal{V}] \in \Gamma(\mathcal{D}_1 \oplus \mathcal{D}_2)$. Namely, the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2$ is integrable. □

Theorem 6.3. *Let $\mathcal{M}$ be a semi-slant submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ endowed with Schouten-van Kampen connection. The distribution $\mathcal{D}_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$\sigma(\mathcal{U}, \phi\mathcal{V}) = \sigma(\phi\mathcal{U}, \mathcal{V}). \tag{6.1}$$

Proof. Now using (2.3) and (5.1), we have

$$\begin{aligned}\phi([\mathcal{U},\mathcal{V}]) &= \phi(\nabla_{\mathcal{U}}\mathcal{V} - \nabla_{\mathcal{V}}\mathcal{U}),\\ &= \phi(\overline{\nabla}_{\mathcal{U}}\mathcal{V} + \sigma(\mathcal{U},\mathcal{V}) - \overline{\nabla}_{\mathcal{V}}\mathcal{U} - \sigma(\mathcal{U},\mathcal{V})),\\ &= \phi\widetilde{\overline{\nabla}}_{\mathcal{U}}\mathcal{V} - g(\mathcal{U},\mathcal{V})\phi\xi + \eta(\mathcal{V})\phi\mathcal{U} - \phi\widetilde{\overline{\nabla}}_{\mathcal{V}}\mathcal{U} + g(\mathcal{V},\mathcal{U})\phi\xi - \eta(\mathcal{U})\phi\mathcal{V}.\end{aligned}$$

From (5.3)

$$\begin{aligned}\phi([\mathcal{U},\mathcal{V}]) &= \widetilde{\overline{\nabla}}_{\mathcal{U}}\phi\mathcal{V} - (\widetilde{\overline{\nabla}}_{\mathcal{U}}\phi)\mathcal{V} + \eta(\mathcal{V})\phi\mathcal{U} + (\widetilde{\overline{\nabla}}_{\mathcal{V}}\phi)\mathcal{U} - \widetilde{\overline{\nabla}}_{\mathcal{V}}\phi\mathcal{U} - \eta(\mathcal{U})\phi\mathcal{V},\\ &= \overset{*}{\nabla}_{\mathcal{U}}\phi\mathcal{V} - \overset{*}{\nabla}_{\mathcal{V}}\phi\mathcal{U} + \eta(\mathcal{V})\phi\mathcal{U} - \eta(\mathcal{U})\phi\mathcal{V} - \sigma(\mathcal{U},\phi\mathcal{V}) + \sigma(\mathcal{V},\phi\mathcal{U}).\end{aligned}$$

In the last equation $\sigma(\mathcal{U},\phi\mathcal{V}) - \sigma(\mathcal{V},\phi\mathcal{U})$ is normal part of $\phi([\mathcal{U},\mathcal{V}])$. In this case if $[\mathcal{U},\mathcal{V}] \in \Gamma(\mathcal{D}_1 \oplus \langle\xi\rangle)$, then $\phi([\mathcal{U},\mathcal{V}]) = 0$. Thus, $\sigma(\mathcal{U},\phi\mathcal{V}) - \sigma(\mathcal{V},\phi\mathcal{U}) = 0$. Hence, we have

$$\sigma(\mathcal{U},\phi\mathcal{V}) = \sigma(\mathcal{V},\phi\mathcal{U}).$$

□

Theorem 6.4. *Let $\mathcal{M}$ be a semi-slant submanifold of a Kenmotsu manifold $\bar{\mathcal{M}}$ endowed with Schouten-van Kampen connection. The distribution $\mathcal{D}_2 \oplus \langle\xi\rangle$ is integrable if and only if*

$$A_{\phi\mathcal{U}}\mathcal{V} = A_{\phi\mathcal{V}}\mathcal{U}, \tag{6.2}$$

for all $\mathcal{U},\mathcal{V} \in \Gamma(\mathcal{D}_2 \oplus \langle\xi\rangle)$.

Proof. From (5.1), for any $\mathcal{U},\mathcal{V} \in \Gamma(\mathcal{D}_2)$

$$\begin{aligned}g([\mathcal{U},\mathcal{V}],\xi) &= g(\overline{\nabla}_{\mathcal{U}}\mathcal{V},\xi) - g(\overline{\nabla}_{\mathcal{V}}\mathcal{U},\xi),\\ &= g(\widetilde{\overline{\nabla}}_{\mathcal{U}}\mathcal{V},\xi) - g(\mathcal{U},\mathcal{V})g(\xi,\xi) + \eta(\mathcal{V})g(\mathcal{U},\xi)\\ &\quad - g(\widetilde{\overline{\nabla}}_{\mathcal{V}}\mathcal{U},\xi) + g(\mathcal{V},\mathcal{U}) - \eta(\mathcal{U})g(\mathcal{V},\xi),\\ &= -g(\widetilde{\overline{\nabla}}_{\mathcal{U}}\xi,\mathcal{V}) + g(\widetilde{\overline{\nabla}}_{\mathcal{V}}\xi,\mathcal{U}) = 0.\end{aligned}$$

Moreover, using (2.3) and (5.1), we have

$$\begin{aligned}\phi([\mathcal{U},\mathcal{V}]) &= \phi(\nabla_{\mathcal{U}}\mathcal{V} - \nabla_{\mathcal{V}}\mathcal{U}),\\ &= \phi(\overline{\nabla}_{\mathcal{U}}\mathcal{V} + \sigma(\mathcal{U},\mathcal{V}) - \overline{\nabla}_{\mathcal{V}}\mathcal{U} - \sigma(\mathcal{V},\mathcal{U})),\\ &= \phi\widetilde{\overline{\nabla}}_{\mathcal{U}}\mathcal{V} + \eta(\mathcal{V})\phi\mathcal{U} - \phi\widetilde{\overline{\nabla}}_{\mathcal{V}}\mathcal{U} - \eta(\mathcal{U})\phi\mathcal{V},\\ &= \widetilde{\overline{\nabla}}_{\mathcal{U}}\phi\mathcal{V} - (\widetilde{\overline{\nabla}}_{\mathcal{U}}\phi)\mathcal{V} + \eta(\mathcal{V})\phi\mathcal{U} + (\widetilde{\overline{\nabla}}_{\mathcal{V}}\phi)\mathcal{U} - \widetilde{\overline{\nabla}}_{\mathcal{V}}\phi\mathcal{U} - \eta(\mathcal{U})\phi\mathcal{V}.\end{aligned}$$

From (5.4), we get

$$\phi([\mathcal{U},\mathcal{V}]) = -A_{\phi\mathcal{V}} + A_{\phi\mathcal{U}} + \overset{*}{\nabla}^{\perp}_{\mathcal{U}}\phi\mathcal{V} + \eta(\mathcal{V})\phi\mathcal{U} - \overset{*}{\nabla}^{\perp}_{\mathcal{V}}\phi\mathcal{U} - \eta(\mathcal{U})\phi\mathcal{V}.$$

In the last equation $A_{\phi\mathcal{U}}\mathcal{V} - A_{\phi\mathcal{V}}\mathcal{U}$ is tangent part of $\phi([\mathcal{U},\mathcal{V}])$. In this case if $[\mathcal{U},\mathcal{V}] \in \Gamma(\mathcal{D}_2 \oplus \langle\xi\rangle)$, then $\phi([\mathcal{U},\mathcal{V}]) = 0$. Thus, $A_{\phi\mathcal{U}}\mathcal{V} - A_{\phi\mathcal{V}}\mathcal{U} = 0$. Then, we have

$$A_{\phi\mathcal{U}}\mathcal{V} = A_{\phi\mathcal{V}}\mathcal{U}.$$

□

Example 6.5. Consider the 5-dimensional proper semi-slant submanifold with the slant angle $\theta \in (0, \pi/2)$ of $\mathbb{R}^7$ defined by

$$\mathcal{U}(k, s, u, v, w) = (k, 0, u, s, v\sin\theta, v\cos\theta, w)$$

with its usual Kenmotsu structure $\mathbb{R}^7 = \mathbb{C}^3 \times \mathbb{R}$,

$$\eta = dt, \quad \xi = \frac{\partial}{\partial t},$$
$$g = \eta \otimes \eta + e^{2t}\left(\sum_{i=1}^{3} dx^i \otimes dx^i + dy^i \otimes dy^i\right),$$
$$\phi_0\left(\sum_{i=1}^{3}\left(\mathcal{U}_i\frac{\partial}{\partial x^i} + \mathcal{V}_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial t}\right) = \sum_{i=1}^{3}\left(-\mathcal{V}_i\frac{\partial}{\partial x^i} + \mathcal{U}_i\frac{\partial}{\partial y^i}\right),$$

where (x^i, y^i, t) $i = 1, 2, 3$ are Cartesian coordinates. Let $\mathcal{M}$ be the above semi-slant submanifold. Then, the tangent space $\mathcal{T}\mathcal{M}$ is spanned by the vectors

$$e_1 = \frac{1}{e^t}\frac{\partial}{\partial x_1}, \qquad e_2 = \frac{1}{e^t}\frac{\partial}{\partial y_1},$$
$$e_3 = \frac{1}{e^t}\frac{\partial}{\partial x_3}, \qquad e_4 = \frac{1}{e^t}\left(\sin\theta\frac{\partial}{\partial y_2} + \cos\theta\frac{\partial}{\partial y_3}\right),$$
$$e_5 = \xi.$$

Clearly, we have

$$\phi e_1 = -\frac{1}{e^t}\frac{\partial}{\partial y_1} = -e_2,$$
$$\phi e_2 = \frac{1}{e^t}\frac{\partial}{\partial x_1} = e_1,$$
$$\phi e_3 = -\frac{1}{e^t}\frac{\partial}{\partial y_3} = -e_3,$$
$$\phi e_4 = \frac{1}{e^t}\left(\sin\theta\frac{\partial}{\partial x_2} + \cos\theta\frac{\partial}{\partial x_3}\right) = e_4,$$
$$\phi e_5 = 0.$$

From these relations, it is evident that $\mathcal{D}_1 = \text{span}\{e_1, e_2\}$ is an invariant distribution and $\mathcal{D}_2 = \text{span}\{e_3, e_4\}$ is a slant distribution with slant angle θ. That is, $\mathcal{M}$ is a 5-dimensional proper semi-slant submanifold of $\mathbb{R}^7$ with its usual Kenmotsu structure. By direct calculations, we obtain

$$[e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1,$$
$$[e_2, e_3] = 0, [e_2, e_4] = 0, [e_2, e_5] = e_2,$$
$$[e_3, e_4] = 0, [e_3, e_5] = e_3, [e_4, e_5] = e_4.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul formula which is

$$\begin{aligned} 2g(\nabla_{\mathcal{U}}\mathcal{V}, \mathcal{Z}) &= \mathcal{U}g(\mathcal{V}, \mathcal{Z}) + \mathcal{V}g(\mathcal{Z}, \mathcal{U}) - \mathcal{Z}g(\mathcal{U}, \mathcal{V}) \\ &\quad - g(\mathcal{U}, [\mathcal{V}, \mathcal{Z}]) - g(\mathcal{V}, [\mathcal{U}, \mathcal{Z}]) + g(\mathcal{Z}, [\mathcal{U}, \mathcal{V}]). \end{aligned} \tag{6.3}$$

Using (6.3), we have

$$g\left(\bar{\nabla}_{e_1}e_1, e_1\right) = 0.$$

Hence, $\bar{\nabla}_{e_1}e_1 = -\xi$. Similarly,

$$\bar{\nabla}_{e_1}e_2 = 0, \bar{\nabla}_{e_1}e_3 = 0, \bar{\nabla}_{e_1}e_4 = 0, \bar{\nabla}_{e_1}e_5 = e_1,$$
$$\bar{\nabla}_{e_2}e_1 = 0, \bar{\nabla}_{e_2}e_2 = -\xi, \bar{\nabla}_{e_2}e_3 = 0, \bar{\nabla}_{e_2}e_4 = 0, \bar{\nabla}_{e_2}e_5 = e_2,$$
$$\bar{\nabla}_{e_3}e_1 = 0, \bar{\nabla}_{e_3}e_2 = 0, \bar{\nabla}_{e_3}e_3 = -\xi, \bar{\nabla}_{e_3}e_4 = 0, \bar{\nabla}_{e_3}e_5 = e_3,$$
$$\bar{\nabla}_{e_4}e_1 = -\xi, \bar{\nabla}_{e_4}e_2 = 0, \bar{\nabla}_{e_4}e_3 = 0, \bar{\nabla}_{e_4}e_4 = -\xi, \bar{\nabla}_{e_4}e_5 = e_4,$$

$$\bar{\nabla}_{e_5} e_1 = 0, \bar{\nabla}_{e_5} e_2 = 0, \bar{\nabla}_{e_5} e_3 = 0, \bar{\nabla}_{e_5} e_4 = 0, \bar{\nabla}_{e_5} e_5 = 0.$$

Moreover, the mean curvature vector is

$$\mathcal{H} = \frac{1}{\dim \mathcal{M}} trace\,(\sigma) = \frac{1}{5}\left(\sigma(e_1, e_1) + \sigma(e_2, e_2) + \sigma(e_3, e_3) + \sigma(e_4, e_4) + \sigma(\xi, \xi)\right) = 0.$$

Hence, $\mathcal{M}$ is a totally geodesic and minimal semi-slant submanifold.

Now, the Schouten-van Kampen connection on $\mathcal{M}$ is given by

$$\begin{aligned}
&\widetilde{\bar{\nabla}}_{e_1} e_1 = 0, \widetilde{\bar{\nabla}}_{e_1} e_2 = 0, \widetilde{\bar{\nabla}}_{e_1} e_3 = 0, \widetilde{\bar{\nabla}}_{e_1} e_4 = 0, \widetilde{\bar{\nabla}}_{e_1} e_5 = 0,\\
&\widetilde{\bar{\nabla}}_{e_2} e_1 = 0, \widetilde{\bar{\nabla}}_{e_2} e_2 = 0, \widetilde{\bar{\nabla}}_{e_2} e_3 = 0, \widetilde{\bar{\nabla}}_{e_2} e_4 = 0, \widetilde{\bar{\nabla}}_{e_2} e_5 = 0,\\
&\widetilde{\bar{\nabla}}_{e_3} e_1 = 0, \widetilde{\bar{\nabla}}_{e_3} e_2 = 0, \widetilde{\bar{\nabla}}_{e_3} e_3 = 0, \widetilde{\bar{\nabla}}_{e_3} e_4 = 0, \widetilde{\bar{\nabla}}_{e_3} e_5 = 0,\\
&\widetilde{\bar{\nabla}}_{e_4} e_1 = 0, \bar{\nabla}_{e_4} e_2 = 0, \widetilde{\bar{\nabla}}_{e_4} e_3 = 0, \widetilde{\bar{\nabla}}_{e_4} e_4 = 0, \widetilde{\bar{\nabla}}_{e_4} e_5 = 0,\\
&\widetilde{\bar{\nabla}}_{e_5} e_1 = 0, \widetilde{\bar{\nabla}}_{e_5} e_2 = 0, \widetilde{\bar{\nabla}}_{e_5} e_3 = 0, \widetilde{\bar{\nabla}}_{e_5} e_4 = 0, \widetilde{\bar{\nabla}}_{e_5} e_5 = 0,
\end{aligned}$$

$$g\left(\sigma\left(e_i, e_j\right), \mathcal{W}\right) = g\left(A_{\mathcal{W}}(e_i), e_j\right) = -g\left(\widetilde{\bar{\nabla}} e_i \mathcal{W}, e_j\right) = g\left(\mathcal{W}, \widetilde{\bar{\nabla}} e_i e_j\right).$$

Moreover, the mean curvature vector is

$$\mathcal{H} = \frac{1}{\dim \mathcal{M}} trace\,(\sigma) = \frac{1}{5}\left(\sigma(e_1, e_1) + \sigma(e_2, e_2) + \sigma(e_3, e_3) + \sigma(e_4, e_4) + \sigma(\xi, \xi)\right) = 0.$$

Hence, $\mathcal{M}$ is totally geodesic and minimal with respect to the Schouten-van Kampen connection on $\bar{\mathcal{M}}$. On the other hand

$$\begin{aligned}
&\sigma(e_1, \phi e_2) = \sigma(e_1, e_1) = 0,\\
&\sigma(\phi e_1, e_2) = \sigma(-e_2, e_2) = -\sigma(e_2, e_2) = 0,\\
&\sigma(e_1, \phi e_2) = \sigma(\phi e_1, e_2).
\end{aligned}$$

Thus, we can see that the equality (6.1) is achieved. And so $\mathcal{D}_1 \oplus \langle \xi \rangle$is integrable.

Also,

$$\begin{aligned}
g\left(A_{\phi e_3} e_4, e_3\right) &= g\left(\sigma(e_4, e_3), \phi e_3\right) = 0,\\
g\left(A_{\phi e_4} e_3, e_3\right) &= g\left(\sigma(e_3, e_3), \phi e_4\right) = 0,\\
g\left(A_{\phi e_3} e_4, e_4\right) &= g\left(\sigma(e_4, e_4), \phi e_3\right) = 0,\\
g\left(A_{\phi e_4} e_3, e_4\right) &= g\left(\sigma(e_3, e_4), \phi e_4\right) = 0,\\
A_{\phi e_3} e_4 &= A_{\phi e_4} e_3,
\end{aligned}$$

Hence, we can see that the equality (6.2) is achieved. So, $\mathcal{D}_2 \oplus \langle \xi \rangle$ is integrable.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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